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The Theory of Saturn's Rings

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V—The Theory of Saturn's Rings

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Introduction

In his Adams Prize Essay for the year 1856 J. Clerk Maxwell* took the first step towards a comprehensive theory of the unique system of rings associated with Saturn.

The main theme of Maxwell's work was to devise a model for the system of rings which could move in a steady state of motion round Saturn and which could be stable for small disturbances, provided the law of gravitation was applicable to the Saturnian system.

Maxwell considered the case a single ring, concentric with Saturn. He concluded that—

(i) the steady motion of a uniform rigid circular ring would be unstable ; and that the steady motion of a non-uniform rigid circular ring could be stable only if the density were very irregular,

(ii) the steady motion of a ring composed of discrete particles, of equal mass, moving in a circle round the planet would be stable if a certain condition were satisfied, and

(iii) the steady motion of a liquid ring would be unstable.

Maxwell therefore concluded that the ring system was composed of discrete particles, *i.e.*, of meteorites. Maxwell's theory has received support from spectroscopic observations and is now generally accepted.

The divisions in the system of rings had to be accounted for. Suggestions had been made that the satellites of Saturn were responsible for zones of instability around the planet. On the basis of Maxwell's theory, Goldsbrough† investigated

* 'Scientific Papers,' vol. I, pp. 288–376.

† 'Phil. Trans.,' A, vol. 222, p. 101 (1921); 'Proc. Roy. Soc.,' A, vol. 101, p. 280 (1922); 'Proc. Roy. Soc.,' A, vol. 106, p. 526 (1924), referred to as I, II, and III respectively.

the problem in two papers, the first being the fundamental paper. He assumed that the system of rings did not affect the motions of the satellites relative to the centre of the planet, so that the problem became an extension of the "Restricted Problem of Three Bodies." In the first paper he considered the perturbations of a single ring of particles by a satellite moving relative to the planet in a circle in the plane of the ring. In the second paper he investigated the perturbations of a single ring of particles by a satellite moving relative to the planet in a circular orbit slightly inclined to the plane of the ring. The effects of each satellite were investigated independently of the possible effects of other satellites. He attributed the divisions to several satellites specifically, and in many respects the results of his theory compare very favourably with the observed data.

In the opinion of the writer the theory as hitherto accepted requires revision on several grounds. In this paper the stability of the steady motion of a single ring of Saturn and the perturbations of it by a satellite moving relative to the centre of Saturn in a coplanar circle will be considered.* The reasons why MAXWELL's investigation appears to the writer to be inconclusive are explained below in § 4.

The investigation is carried out according to the usual method of "first approximations." It is doubtful whether agreement on the vexed question of the definitions of the terms "stability" and "instability" has been reached. Consequently, it is well to remember that the so-called method of first approximations may lead to erroneous conclusions; a very interesting example in support of the above statement has been furnished by CHERRY.†

The astronomical unit of mass is chosen. Saturn is assumed to be a homogeneous sphere of mass M, and its centre is denoted by O. A ring is supposed to consist of a set of discrete particles, each of which has the mass m, n in number.

The suffixes λ and μ are used to characterize the particles: $\lambda, \mu = 1, \dots, n$. The summation sign $\underset{\mu}{\Sigma}$ is used to denote summation over all the particles; and the summation sign $\underset{\mu}{\Sigma'}$ is used to denote summation over all the particles except for $\mu = \lambda$.

I shall prove at this stage a lemma which will be used later.

Lemma—Let α be a variable assuming the values $1, \dots, n-1$; and let a function $\psi(\alpha)$ be defined for all the values of α. Further, let the function $\psi(-\alpha)$ be defined for all the values of α, and let $\psi(n-\alpha) = \psi(-\alpha)$. Then

$\underset{\mu}{\Sigma'}\, \psi(\mu-\lambda)\, e^{\frac{2\pi i s(\mu-\lambda)}{n}}$, (where $s = 1, \dots, n$), is independent of λ, and is equal to $\sum_{\alpha=1}^{n-1} \psi(\alpha)\, e^{\frac{2\pi i s\alpha}{n}}$.

* In a short paper ('Phil. Mag.,' vol. 16, p. 575 (1933)), the writer ventured to criticize the first part of MAXWELL's essay.

† 'Trans. Camb. Phil. Soc.,' vol. 23, p. 199 (1925). *Cf.* WHITTAKER, 'Analytical Dynamics,' 3 ed. (1927), § 182, p. 412.

Let $1 < \lambda < n$ (of course applicable when $n \geqslant 3$). Then

$$\Sigma'_{\mu} \psi(\mu-\lambda)\, e^{\frac{2\pi is(\mu-\lambda)}{n}} = \sum_{\mu=1}^{\lambda-1} \psi(\mu-\lambda)\, e^{\frac{2\pi is(\mu-\lambda)}{n}} + \sum_{\mu=\lambda+1}^{n} \psi(\mu-\lambda)\, e^{\frac{2\pi is(\mu-\lambda)}{n}}$$

$$= \psi(1)\, e^{\frac{2\pi is\times 1}{n}} + \psi(2)\, e^{\frac{2\pi is\times 2}{n}} + \ldots + \psi(n-\lambda)\, e^{\frac{2\pi is(n-\lambda)}{n}}$$

$$+ \psi(1-\lambda)\, e^{\frac{2\pi is(1-\lambda)}{n}} + \psi(2-\lambda)\, e^{\frac{2\pi is(2-\lambda)}{n}} + \ldots + \psi(\lambda-1-\lambda)\, e^{\frac{2\pi is(\lambda-1-\lambda)}{n}}$$

$$= \psi(1)\, e^{\frac{2\pi is\times 1}{n}} + \psi(2)\, e^{\frac{2\pi is\times 2}{n}} + \ldots + \psi(n-\lambda)\, e^{\frac{2\pi is(n-\lambda)}{n}}$$

$$+ \psi(n-\overline{\lambda-1})\, e^{\frac{2\pi is(n-\lambda+1)}{n}} + \psi(n-\overline{\lambda-2})\, e^{\frac{2\pi is(n-\lambda+2)}{n}} + \ldots + \psi(n-1)\, e^{\frac{2\pi is(n-1)}{n}}$$

$$= \sum_{\alpha=1}^{n-1} \psi(\alpha)\, e^{\frac{2\pi is\alpha}{n}}.$$

Similarly for the extreme cases $\lambda = 1$, $\lambda = n$, we have respectively

$$\Sigma'_{\mu} \psi(\mu-\lambda)\, e^{\frac{2\pi is(\mu-\lambda)}{n}} = \sum_{\alpha=1}^{n-1} \psi(\alpha)\, e^{\frac{2\pi is\alpha}{n}}, \quad \text{and} \quad \Sigma'_{\mu} \psi(\mu-\lambda)\, e^{\frac{2\pi is(\mu-\lambda)}{n}} = \sum_{\alpha=1}^{n-1} \psi(\alpha)\, e^{\frac{2\pi is\alpha}{n}}.$$

Hence we have, for all values of λ $(= 1, \ldots, n)$,

$$\Sigma'_{\mu} \psi(\mu-\lambda)\, e^{\frac{2\pi is(\mu-\lambda)}{n}} = \sum_{\alpha=1}^{n-1} \psi(\alpha)\, e^{\frac{2\pi is\alpha}{n}}, \text{ which is independent of } \lambda.$$

Similarly,

$$\Sigma'_{\mu} \psi(\mu-\lambda)\, e^{-\frac{2\pi is(\mu-\lambda)}{n}} = \sum_{\alpha=1}^{n-1} \psi(\alpha)\, e^{-\frac{2\pi is\alpha}{n}}, \text{ which is independent of } \lambda.$$

Further,

$$\Sigma'_{\mu} \psi(\mu-\lambda) = \sum_{\alpha=1}^{n-1} \psi(\alpha), \text{ which is independent of } \lambda.$$

PART I

The Stability of the Steady Motion of a Single Ring

§1. *The equations of the disturbed motion of a set of* n (>1) *discrete particles, each of mass* m, *forming a ring round a central homogeneous spherical body of mass* M.

Let Ox, Oy, Oz be "fixed" axes through O, the centre of the spherical body. The motion of the particles relative to O is considered. Let x_λ, y_λ, z_λ, be the Cartesian co-ordinates and r_λ, θ_λ, z_λ the cylindrical co-ordinates of the particle λ relative to the frame Oxyz : $x_\lambda = r_\lambda \cos\theta_\lambda$, $y_\lambda = r_\lambda \sin\theta_\lambda$.

Let*

$$\mathrm{F}_\lambda = \frac{(\mathrm{M}+m)}{(x_\lambda{}^2+y_\lambda{}^2+z_\lambda{}^2)^{\frac{1}{2}}} + \sum_\mu{}' m \left[\frac{1}{\{(x_\lambda - x_\mu)^2 + (y_\lambda - y_\mu)^2 + (z_\lambda - z_\mu)^2\}^{\frac{1}{2}}} - \frac{x_\lambda x_\mu + y_\lambda y_\mu + z_\lambda z_\mu}{(x^2_\mu + y_\mu{}^2 + z_\mu{}^2)^{3/2}}\right]$$

$$= \frac{(\mathrm{M}+m)}{(r_\lambda{}^2+z_\lambda{}^2)^{\frac{1}{2}}} + \sum_\mu{}' m \left[\frac{1}{\{r_\lambda{}^2 - 2\, r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu{}^2 + (z_\lambda - z_\mu)^2\}^{\frac{1}{2}}} - \frac{r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + z_\lambda z_\mu}{(r_\mu{}^2 + z_\mu{}^2)^{3/2}}\right].$$

For the purposes of mathematical investigation it is assumed that the central body and the system of particles constitute an " isolated system " and that, relative to Newtonian axes, the motion of each member of the whole system is described by the law of gravitation. Then, according to the methods of dynamical astronomy, the equations of motion of the particle λ relative to O are—

$$\ddot{x}_\lambda = \frac{\partial \mathrm{F}_\lambda}{\partial x_\lambda}, \quad \ddot{y}_\lambda = \frac{\partial \mathrm{F}_\lambda}{\partial y_\lambda}, \quad \ddot{z}_\lambda = \frac{\partial \mathrm{F}_\lambda}{\partial z_\lambda},$$

or, in terms of the cylindrical polar co-ordinates,

$$\ddot{r}_\lambda - r_\lambda \dot{\theta}_\lambda{}^2 = \frac{\partial \mathrm{F}_\lambda}{\partial r_\lambda}, \quad 2\dot{r}_\lambda \dot{\theta}_\lambda + r_\lambda \ddot{\theta}_\lambda = \frac{1}{r_\lambda}\frac{\partial \mathrm{F}_\lambda}{\partial \theta_\lambda}, \quad \ddot{z}_\lambda = \frac{\partial \mathrm{F}_\lambda}{\partial z_\lambda}.$$

It is now assumed that initially the system of particles is in a state of steady motion relative to O. In the steady state the particles are assumed to form the vertices of a regular polygon, inscribed in a circle centre O and radius a, rotating with uniform angular velocity ω. This steady state is called the " datum " state or configuration and the values belonging to that state will be distinguished by the suffix " 0."

The planes of reference are chosen in such a manner that the plane of the ring in the datum state coincides with the xy plane. Then

$$(r_\lambda)_0 = a\,, \ (\theta_\lambda)_0 = \omega t + \frac{2\pi\lambda}{n} + \varepsilon\,, \ (z_\lambda)_0 = 0, \ \varepsilon \text{ being an arbitrary constant.}$$

The datum state is dynamically possible without any appeal to approximations provided ω is appropriately chosen†; that is to say, $r_\lambda = a$, $\theta_\lambda = \omega t + \frac{2\pi\lambda}{n} + \varepsilon$,

* *Cf.* F. Tisserand, ' Traité de mécanique céléste,' t. I, 1889, chapter iii, p. 75.

† In the opinion of the writer the exact dynamical possibility of the datum state is desirable for the following reason. The present investigation, like other similar investigations, is only an approximation. Should the datum state not be actually possible, there will be errors present from the very beginning ; and owing to the insolubility of the general problem it will not be possible to separate the " genuine " disturbance from the " inherent " errors when all the quantities concerned are small. It seems to the writer that the present datum state will not be possible unless the particles are all of the same mass as the corresponding conditions are not satisfied. The desideratum restricts severely the suitable models. Such a restriction reflects rather on our mathematical resources than on the actual composition of the ring system.

$z_\lambda = 0$ $(\lambda = 1, \ldots, n)$ is an exact solution of the equations of motion. The necessary and sufficient conditions are $\left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 = -a\omega^2$, $\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)_0 = 0$, $\left(\frac{\partial F_\lambda}{\partial z_\lambda}\right)_0 = 0$ $(\lambda = 1, \ldots, n)$. Since ω is independent of λ, the condition $\left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 = -a\omega^2$ amounts to requiring that $\left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0$ should have the same value for all values of λ and that this value should be negative; then ω^2 would be determined in terms of M, m, a, and n. It will be shown in § 2 that

$$\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)_0 = 0,\ \left(\frac{\partial F_\lambda}{\partial z_\lambda}\right)_0 = 0,\ \left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 = -\frac{1}{a^2}\left\{M + \tfrac{1}{4}m\sum_{\alpha=1}^{n-1}\frac{1}{\sin\frac{\pi\alpha}{n}}\right\} = -a\omega^2.$$

Now let the system of particles be disturbed in the initial plane of the ring; and in the disturbed state, let

$$r_\lambda = a(1+\rho_\lambda),\ \theta_\lambda = (\theta_\lambda)_0 + \sigma_\lambda = \omega t + \frac{2\pi\lambda}{n} + \varepsilon + \sigma_\lambda,\ z_\lambda = 0.$$

It is further assumed that the ρ_λ's, the σ_λ's and their derivatives are so small that their products may be neglected in the subsequent work. Then the equations of motion of the particle λ are—

$$a\left\{\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda - \omega^2\rho_\lambda - \omega^2\right\} = \left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 + \sum_\mu a\rho_\mu\left\{\frac{\partial^2 F_\lambda}{\partial r_\mu \partial r_\lambda}\right\}_0 + \sum_\mu \sigma_\mu\left\{\frac{\partial^2 F_\lambda}{\partial \theta_\mu \partial r_\lambda}\right\}_0 \quad . \quad . \quad (1.1)$$

$$a(\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda) = \left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)_0 + \sum_\mu a\rho_\mu\left\{\frac{\partial}{\partial r_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)\right\}_0 + \sum_\mu \sigma_\mu\left\{\frac{\partial}{\partial \theta_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)\right\}_0 \quad . \quad . \quad . \quad (1.2)$$

§ 2. *The equations of the disturbed motion of the particle* λ

$$\frac{\partial F_\lambda}{\partial z_\lambda} = -\frac{(M+m)z_\lambda}{(r_\lambda^2 + z_\lambda^2)^{3/2}} + \sum_\mu{}' m\left[\frac{(z_\mu - z_\lambda)}{\{(x_\lambda - x_\mu)^2 + (y_\lambda - y_\mu)^2 + (z_\lambda - z_\mu)^2\}^{3/2}} - \frac{z_\mu}{(r_\mu^2 + z_\mu^2)^{3/2}}\right]$$

therefore,

$$\left(\frac{\partial F_\lambda}{\partial z_\lambda}\right)_0 = 0.$$

Since $z_\lambda = 0$ for all values of λ in the disturbed motion, it would be more convenient to take

$$F_\lambda = \frac{(M+m)}{r_\lambda} + \sum_\mu{}' m\left[\frac{1}{\{r_\lambda^2 - 2r_\lambda r_\mu\cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{1/2}} - \frac{r_\lambda\cos(\theta_\lambda - \theta_\mu)}{r_\mu^2}\right].$$

$$\frac{\partial F_\lambda}{\partial r_\lambda} = -\frac{(M+m)}{r_\lambda^2} - \sum_\mu{}' m\left[\frac{r_\lambda - r_\mu\cos(\theta_\lambda - \theta_\mu)}{\{r_\lambda^2 - 2r_\lambda r_\mu\cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} + \frac{\cos(\theta_\lambda - \theta_\mu)}{r_\mu^2}\right]$$

$$\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda} = \sum_\mu{}' m\left[\frac{r_\mu}{\{r_\lambda^2 - 2r_\lambda r_\mu\cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} - \frac{1}{r_\mu^2}\right]\sin(\theta_\mu - \theta_\lambda)$$

For $\mu \neq \lambda$

$$\frac{\partial^2 F_\lambda}{\partial r_\mu \partial r_\lambda} = -m\left[-\frac{\cos(\theta_\lambda - \theta_\mu)}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} - \frac{3\{r_\lambda - r_\mu \cos(\theta_\lambda - \theta_\mu)\}\{r_\mu - r_\lambda \cos(\theta_\lambda - \theta_\mu)\}}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{5/2}} - \frac{2\cos(\theta_\lambda - \theta_\mu)}{r_\mu^3}\right]$$

$$\frac{\partial^2 F_\lambda}{\partial r_\lambda^2} = \frac{2(M+m)}{r_\lambda^3} - \sum_\mu{}' m\left[\frac{1}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} - \frac{3\{r_\lambda - r_\mu \cos(\theta_\lambda - \theta_\mu)\}^2}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{5/2}}\right]$$

$$\sum_\mu \frac{\partial^2 F_\lambda}{\partial \theta_\mu \partial r_\lambda}\sigma_\mu = \sum_\mu{}' m\left[-\frac{r_\mu}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} + \frac{3r_\lambda r_\mu\{r_\lambda - r_\mu \cos(\theta_\lambda - \theta_\mu)\}}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{5/2}} + \frac{1}{r_\mu^2}\right]\sin(\theta_\mu - \theta_\lambda)(\sigma_\mu - \sigma_\lambda)$$

For $\mu \neq \lambda$

$$\frac{\partial}{\partial r_\mu}\left\{\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right\} = m\left[\frac{1}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} - \frac{3r_\mu\{r_\mu - r_\lambda \cos(\theta_\lambda - \theta_\mu)\}}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{5/2}} + \frac{2}{r_\mu^3}\right]\sin(\theta_\mu - \theta_\lambda)$$

$$\frac{\partial}{\partial r_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right) = \sum_\mu{}' m\left[-\frac{3r_\mu\{r_\lambda - r_\mu \cos(\theta_\lambda - \theta_\mu)\}}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{5/2}}\right]\sin(\theta_\mu - \theta_\lambda)$$

$$\sum_\mu \frac{\partial}{\partial \theta_\mu}\left\{\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right\}\sigma_\mu = \sum_\mu{}' m\left[\frac{r_\mu}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{3/2}} - \frac{1}{r_\mu^2}\right]\cos(\theta_\mu - \theta_\lambda)(\sigma_\mu - \sigma_\lambda) - \sum_\mu{}' m\left[\frac{3r_\lambda r_\mu^2}{\{r_\lambda^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{5/2}}\sin^2(\theta_\mu - \theta_\lambda)\right](\sigma_\mu - \sigma_\lambda).$$

$$\left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 = -\frac{(M+m)}{a^2} - \sum_\mu{}' \frac{m}{a^2}\left[\frac{1}{4\sin(\mu \sim \lambda)\pi/n} + \cos(\mu - \lambda)2\pi/n\right]$$
$$= -\frac{1}{a^2}\left\{M + \tfrac{1}{4}m\sum_\mu{}' \frac{1}{\sin(\mu \sim \lambda)\pi/n}\right\}$$

by using the relation

$$1 + \sum_\mu{}' \cos(\mu - \lambda)2\pi/n = 0.$$

$\dfrac{1}{\sin (\mu \sim \lambda) \pi/n}$ satisfies all the conditions imposed upon $\psi (\mu - \lambda)$ in the Lemma, and, therefore,

$$\sum_\mu{}' \frac{1}{\sin (\mu \sim \lambda) \pi/n} = \sum_{\alpha=1}^{n-1} \frac{1}{\sin \frac{\pi\alpha}{n}}.$$

Therefore

$$\left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 = -\frac{1}{a^2}\left\{M + \frac{m}{4}\sum_{\alpha=1}^{n-1}\frac{1}{\sin \frac{\pi\alpha}{n}}\right\},$$

which has the same value for all values of λ, and

$$\omega^2 a^3 = M + \frac{m}{4}\sum_{\alpha=1}^{n-1}\frac{1}{\sin \frac{\pi\alpha}{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

Also

$$\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)_0 = \frac{m}{a^2}\sum_\mu{}'\left[\frac{1}{8 \sin^3 (\mu \sim \lambda) \pi/n} - 1\right] \sin (\mu - \lambda) 2\pi/n.$$

But $\left\{\dfrac{1}{8 \sin^3 (\mu \sim \lambda) \pi/n} - 1\right\} \sin (\mu - \lambda) 2\pi/n$ satisfies all the conditions imposed upon $\psi (\mu - \lambda)$ in the Lemma. Hence it follows that

$$\sum_\mu{}'\left\{\frac{1}{8 \sin^3 (\mu \sim \lambda) \pi/n} - 1\right\} \sin (\mu - \lambda) 2\pi/n = \sum_{\alpha=1}^{n-1}\left\{\frac{1}{8 \sin^3 \frac{\pi\alpha}{n}} - 1\right\} \sin \frac{2\pi\alpha}{n} = 0,$$

and

$$\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)_0 = 0.$$

Again,

$$\sum_\mu \left(\frac{\partial^2 F_\lambda}{\partial r_\mu \, \partial r_\lambda}\right)_0 a\rho_\mu = \frac{2 (M + m)}{a^2}\rho_\lambda - \frac{m}{a^2}\rho_\lambda \sum_\mu{}'\left[\frac{1}{8 \sin^3 (\mu \sim \lambda) \pi/n} - \frac{3}{8 \sin (\mu \sim \lambda) \pi/n}\right]$$
$$+ \frac{m}{a^2}\sum_\mu{}'\left[\frac{\cos (\mu - \lambda) 2\pi/n}{8 \sin^3 (\mu \sim \lambda) \pi/n} + \frac{3}{8 \sin (\mu \sim \lambda) \pi/n} + 2 \cos (\mu - \lambda) 2\pi/n\right]\rho_\mu.$$

$\dfrac{1}{8 \sin^3 (\mu \sim \lambda) \pi/n} - \dfrac{3}{8 \sin (\mu \sim \lambda) \pi/n}$ satisfies the conditions imposed on $\psi (\mu - \lambda)$ in the Lemma and, therefore,

$$\sum_\mu{}'\left\{\frac{1}{8 \sin^3 (\mu \sim \lambda) \pi/n} - \frac{3}{8 \sin (\mu \sim \lambda) \pi/n}\right\} = \sum_{\alpha=1}^{n-1}\left[\frac{1}{8 \sin^3 \frac{\pi\alpha}{n}} - \frac{3}{8 \sin \frac{\pi\alpha}{n}}\right].$$

Hence,

$$\sum_{\mu}\left(\frac{\partial^2 F_\lambda}{\partial r_\mu\,\partial r_\lambda}\right)_0 a\rho_\mu = \frac{2\,(M+m)}{a^2}\,\rho_\lambda - \frac{m}{a^2}\,\rho_\lambda \sum_{\alpha=1}^{n-1}\left[\frac{1}{8\sin^3\frac{\pi\alpha}{n}} - \frac{3}{8\sin\frac{\pi\alpha}{n}}\right]$$

$$+\frac{m}{a^2}\sum_{\mu}{}'\left[\frac{\cos(\mu-\lambda)\,2\pi/n + 3\sin^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + 2\cos(\mu-\lambda)\,2\pi/n\right]\rho_\mu$$

$$=\left[2\,\omega^2 + \frac{2m}{a^3} - \frac{m}{8a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1}{\sin^3\frac{\pi\alpha}{n}} + \frac{1}{\sin\frac{\pi\alpha}{n}}\right\}\right]a\rho_\lambda$$

$$+\frac{m}{a^2}\sum_{\mu}{}'\left[\frac{1+\sin^2(\mu-\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + 2\cos(\mu-\lambda)\,2\pi/n\right]\rho_\mu,$$

using equation (1.3).

$$\sum_{\mu}\left(\frac{\partial^2 F_\lambda}{\partial\theta_\mu\,\partial r_\lambda}\right)_0\sigma_\mu = \frac{m}{a^2}\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n\,(\sigma_\mu-\sigma_\lambda)$$

$$=\frac{m}{a^2}\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n\,\sigma_\mu$$

$$-\frac{m}{a^2}\,\sigma_\lambda\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n$$

$\left[\dfrac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n$ satisfies the conditions imposed upon $\psi(\mu-\lambda)$ in the Lemma. Therefore,

$$\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n$$

$$=\sum_{\alpha=1}^{n-1}\left[\frac{1}{16\sin^3\pi\alpha/n}+1\right]\sin 2\pi\alpha/n = 0.$$

Finally,

$$\sum_{\mu}\left(\frac{\partial^2 F_\lambda}{\partial\theta_\mu\,\partial r_\lambda}\right)_0\sigma_\mu = \frac{m}{a^2}\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n\,\sigma_\mu$$

$$\left\{\frac{\partial}{\partial r_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial\theta_\lambda}\right)\right\}_0 = -\frac{3}{16}\frac{m}{a^3}\sum_{\mu}{}'\frac{\sin(\mu-\lambda)\,2\pi/n}{\sin^3(\mu\sim\lambda)\,\pi/n}$$

$= 0$ as $\dfrac{\sin(\mu-\lambda)\,2\pi/n}{\sin^3(\mu\sim\lambda)\,\pi/n}$ satisfies the conditions imposed upon $\psi(\mu-\lambda)$ in the Lemma.

$$\sum_{\mu}' a\,\rho_\mu \left\{\frac{\partial}{\partial r_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)\right\}_0 = \frac{m}{a^2}\sum_{\mu}'\left[2 - \frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}\right]\sin(\mu-\lambda)\,2\pi/n\,\rho_\mu$$

$$\sum_{\mu} \sigma_\mu \left\{\frac{\partial}{\partial \theta_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)\right\}_0$$

$$= \frac{m}{a^2}\sum_{\mu}'\left[\frac{\cos(\mu-\lambda)\,2\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} - \frac{3\cos^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} - \cos(\mu-\lambda)\,2\pi/n\right](\sigma_\mu-\sigma_\lambda)$$

$$= -\frac{m}{a^2}\sum_{\mu}'\left[\frac{1+\cos^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + \cos(\mu-\lambda)\,2\pi/n\right](\sigma_\mu-\sigma_\lambda)$$

$$= \frac{m}{a^2}\sigma_\lambda\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\pi\alpha/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + \cos 2\pi\alpha/n\right]$$

$$-\frac{m}{a^2}\sum_{\mu}'\left[\frac{1+\cos^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + \cos(\mu-\lambda)\,2\pi/n\right]\sigma_\mu,$$

as $\dfrac{1+\cos^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + \cos(\mu-\lambda)\,2\pi/n$ satisfies the conditions imposed on $\psi(\mu-\lambda)$ in the Lemma.

Finally, equations (1·1) and (1·2) for the disturbed motion of the particle λ assume the form—

$$\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda = \left[3\omega^2 + \frac{2m}{a^3} - \frac{m}{8a^3}\sum_{\alpha=1}^{n-1}\left(\frac{1}{\sin\frac{\pi\alpha}{n} + \sin^3\frac{\pi\alpha}{n}}\right)\right]\rho_\lambda$$

$$+\frac{m}{a^3}\sum_{\mu}'\left[\frac{1+\sin^2\frac{(\mu\sim\lambda)\,\pi}{n}}{8\sin^3\frac{(\mu\sim\lambda)\,\pi}{n}} + 2\cos(\mu-\lambda)\,2\pi/n\right]\rho_\mu$$

$$+\frac{m}{a^3}\sum_{\mu}'\left[\frac{1}{16\sin^3\frac{(\mu\sim\lambda)\,\pi}{n}} + 1\right]\sin(\mu-\lambda)\,2\pi/n\,\sigma_\mu \qquad \text{(A. }\lambda)$$

$$\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda = \frac{m}{a^3}\sum_{\mu}'\left[2 - \frac{1}{16\sin^3\frac{(\mu\sim\lambda)\,\pi}{n}}\right]\sin(\mu-\lambda)\,2\pi/n\,\rho_\mu$$

$$+\frac{m}{a^3}\sigma_\lambda\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\frac{\pi\alpha}{n}}{8\sin^3\frac{\pi\alpha}{n}} + \cos\frac{2\pi\alpha}{n}\right] - \frac{m}{a^3}\sum_{\mu}'\left[\frac{1+\cos^2\frac{(\mu\sim\lambda)\,\pi}{n}}{8\sin^3\frac{(\mu\sim\lambda)\,\pi}{n}}\right.$$

$$\left. + \cos(\mu-\lambda)\,2\pi/n\right]\sigma_\mu \qquad \text{(B. }\lambda)$$

We have n interdependent pairs of linear differential equations of the second order for the ρ_λ's and the σ_λ's. The task of integrating the n pairs (A.λ) and (B.λ), as they stand, is hopeless. Hence it is necessary to derive n independent pairs of linear differential equations of the second order from the n interdependent pairs of linear differential equations of the second order (A.λ) and (B.λ) by changing the dependent variables ρ_λ's and σ_λ's. The transformed equations can be integrated completely, at least theoretically.

Such a transformation is possible. In § 3 new dependent variables k_s and l_s ($s = 1, ..., n$), defined as linear functions of the ρ_λ's and σ_λ's respectively are introduced. The k_s's and the l_s's have no physical significance as they will be complex, whereas the original dependent variables, the ρ_λ's and the σ_λ's are real. The ρ_λ's and the σ_λ's will be called "primary" dependent variables and the k_s's and the l_s's will be called "secondary" dependent variables.

§ 3. "*Primary*" *and* "*secondary*" *dependent variables*

Let r and s be variable suffixes ranging from 1 to n. Let

$$\rho_\lambda = \sum_{s=1}^{n} e^{\frac{2\pi i\lambda s}{n}} k_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot1/\lambda)$$

$$\sigma_\lambda = \sum_{s=1}^{n} e^{\frac{2\pi i\lambda s}{n}} l_s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot2/\lambda)$$

where $i = \sqrt{-1}$.

Multiplying both sides of $(4\cdot1/\lambda)$ by $e^{-\frac{2\pi i\lambda r}{n}}$ and summing for λ from 1 to n, we have

$$\sum_{\lambda=1}^{n} \rho_\lambda e^{-\frac{2\pi i\lambda r}{n}} = \sum_{\lambda=1}^{n}\sum_{s=1}^{n} e^{\frac{2\pi i\lambda(s-r)}{n}} k_s$$

$$= \sum_{s=1}^{n}\left[k_s\left\{\sum_{\lambda=1}^{n} e^{\frac{2\pi i\lambda(s-r)}{n}}\right\}\right]$$

by rearrangement.

If $s \neq r$,

$$\sum_{\lambda=1}^{n} e^{\frac{2\pi i\lambda(s-r)}{n}} = e^{\frac{2\pi i(s-r)}{n}} \frac{1-e^{\frac{2\pi i(s-r)}{n}\cdot n}}{1-e^{\frac{2\pi i(s-r)}{n}}}$$

$$= 0 \qquad \text{for } 0 < |s-r| < n \text{ when } r \neq s.$$

If $s = r$,

$$\sum_{\lambda=1}^{n} e^{\frac{2\pi i\lambda(s-r)}{n}} = n.$$

Therefore,

$$\sum_{\lambda=1}^{n} \rho_\lambda e^{-\frac{2\pi i\lambda r}{n}} = nk_r$$

or

$$k_r = \frac{1}{n}\sum_{\lambda=1}^{n} \rho_\lambda e^{-\frac{2\pi i\lambda r}{n}}.$$

The same reasoning applies to the set $(3\cdot2/\lambda)$.

The transformation is, therefore, regular and its inverse is

$$k_s = \frac{1}{n}\sum_{\lambda=1}^{n} \rho_\lambda e^{-\frac{2\pi i\lambda s}{n}}, \quad \text{(3·3/s)}$$

$$l_s = \frac{1}{n}\sum_{\lambda=1}^{n} \sigma_\lambda e^{-\frac{2\pi i\lambda s}{n}}. \quad \text{(3·4/s)}$$

It will be convenient to evaluate some quantities before we turn our attention to the n pairs of equations (A.λ) and (B.λ).

Provided $\mu \neq \lambda$ each of the functions

$$\left[\frac{1+\sin^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n}+2\cos(\mu-\lambda)\,2\pi/n\right],$$

$$\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n,$$

$$\left[2-\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}\right]\sin(\mu-\lambda)\,2\pi/n,$$

and

$$\left[\frac{1+\cos^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n}+\cos(\mu-\lambda)\,2\pi/n\right]$$

satisfy the conditions imposed upon $\psi(\mu-\lambda)$ in the Lemma. Therefore we have the following relations :—

$$\sum_\mu{}'\left[\frac{1+\sin^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n}+2\cos(\mu-\lambda)\,2\pi/n\right]e^{\pm\frac{2\pi is(\mu-\lambda)}{n}}$$

$$=\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos2\pi\alpha/n\right]e^{\pm\frac{2\pi is\alpha}{n}}$$

$$=\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos2\pi\alpha/n\right]\cos\frac{2\pi s\alpha}{n}$$
$$\pm i\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos2\pi\alpha/n\right]\sin\frac{2\pi s\alpha}{n}$$

$$=\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos2\pi\alpha/n\right]\cos\frac{2\pi s\alpha}{n};$$

and consequently

$$\sum_\mu{}'\left[\frac{1+\sin^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n}+2\cos(\mu-\lambda)\,2\pi/n\right]\cos(\mu-\lambda)\,2\pi/n$$

$$=\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos2\pi\alpha/n\right]\cos2\pi s\alpha/n.$$

Similarly,

$$\sum_\mu{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n\;e^{\pm\frac{2\pi is(\mu-\lambda)}{n}}$$

$$=\pm i\sum_{\alpha=1}^{n-1}\left[\frac{1}{16\sin^3\pi\alpha/n}+1\right]\sin2\pi\alpha\sin2\pi s\alpha/n,$$

and

$$\Sigma'_{\mu}\left[\frac{1}{16\sin^3(\mu\sim\lambda)\pi/n}+1\right]\sin(\mu-\lambda)2\pi/n\sin(\mu-\lambda)2\pi s/n$$
$$=\sum_{\alpha=1}^{n-1}\left[\frac{1}{16\sin^3\pi\alpha/n}+1\right]\sin 2\pi\alpha/n\sin 2\pi s\alpha/n\,;$$

$$\Sigma'_{\mu}\left[2-\frac{1}{16\sin^3\dfrac{(\mu\sim\lambda)\pi}{n}}\right]\sin(\mu-\lambda)2\pi/n\,e^{\pm\frac{2\pi i(\mu-\lambda)s}{n}}$$
$$=\pm i\sum_{\alpha=1}^{n-1}\left[2-\frac{1}{16\sin^3\dfrac{\pi\alpha}{n}}\right]\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n},$$

and

$$\Sigma'_{\mu}\left[2-\frac{1}{16\sin^3\dfrac{(\mu\sim\lambda)\pi}{n}}\right]\sin(\mu-\lambda)2\pi/n\sin(\mu-\lambda)2\pi s/n$$
$$=\sum_{\alpha=1}^{n-1}\left[2-\frac{1}{16\sin^3\dfrac{\pi\alpha}{n}}\right]\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}\,;$$

$$\Sigma'_{\mu}\left[\frac{1+\cos^2\dfrac{(\mu\sim\lambda)\pi}{n}}{8\sin^3\dfrac{(\mu\sim\lambda)\pi}{n}}+2\cos(\mu-\lambda)2\pi/n\right]e^{\pm 2\pi i(\mu-\lambda)s/n}$$
$$=\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos 2\pi\alpha/n\right]\cos 2\pi s\alpha/n,$$

and

$$\Sigma'_{\mu}\left[\frac{1+\cos^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+2\cos(\mu-\lambda)2\pi/n\right]\cos(\mu-\lambda)2\pi s/n$$
$$=\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+2\cos 2\pi\alpha/n\right]\cos 2\pi s\alpha/n.$$

$$\frac{1}{n}\sum_{\lambda=1}^{n}e^{-\frac{2\pi i\lambda s}{n}}\left\{\Sigma'_{\mu}\left[\frac{1+\sin^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+2\cos(\mu-\lambda)2\pi/n\right]\rho_\mu\right\}$$
$$=\frac{1}{n}\sum_{\lambda=1}^{n}\left\{\Sigma'_{\mu}\left[\frac{1+\sin^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+2\cos(\mu-\lambda)2\pi/n\right]\rho_\mu e^{-\frac{2\pi i\lambda s}{n}}\right\}$$
$$=\frac{1}{n}\sum_{\lambda=1}^{n}\left\{\Sigma'_{\mu}\left[\frac{1+\sin^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+2\cos(\mu-\lambda)2\pi/n\right]\rho_\mu e^{-\frac{2\pi i\mu s}{n}}e^{-\frac{2\pi i(\lambda-\mu)s}{n}}\right\}$$
$$=\frac{1}{n}\sum_{\lambda=1}^{n}\rho_\lambda e^{-\frac{2\pi i\lambda s}{n}}\left\{\Sigma'_{\mu}\left[\frac{1+\sin^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+2\cos(\mu-\lambda)2\pi/n\right]e^{-\frac{2\pi i(\mu-\lambda)s}{n}}\right\}$$

by rearrangement

$$= \frac{1}{n}\sum_{\lambda=1}^{n} \rho_\lambda e^{-\frac{2\pi i\lambda s}{n}} \left\{\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2 \pi\alpha/n}{8\sin^3 \pi\alpha/n} + 2\cos 2\pi\alpha/n\right]\cos\frac{2\pi s\alpha}{n}\right\}$$

$$= k_s\left\{\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2 \pi\alpha/n}{8\sin^3 \pi\alpha/n} + 2\cos 2\pi\alpha/n\right]\cos\frac{2\pi s\alpha}{n}\right\}.$$

Similarly,

$$\frac{1}{n}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n\,\sigma_\mu\right\}$$

$$= i\left\{\sum_{\alpha=1}^{n-1}\left[\frac{1}{16\sin^3\pi\alpha/n}+1\right]\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}\right\} l_s,$$

$$\frac{1}{n}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\sum_{\mu}{}'\left[2-\frac{1}{16\sin^3(\mu\sim\lambda)\pi/n}\right]\sin(\mu-\lambda)\,2\pi/n\,\rho_\mu\right\}$$

$$= i\left\{\sum_{\alpha=1}^{n-1}\left[2-\frac{1}{16\sin^3\frac{\pi\alpha}{n}}\right]\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}\right\} k_s,$$

and

$$\frac{1}{n}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\sum_{\mu}{}'\left[\frac{1+\cos^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+\cos(\mu-\lambda)\,2\pi/n\right]\sigma_\mu\right\}$$

$$= \left\{\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\pi\alpha/n}{8\sin^3\pi\alpha/n}+\cos 2\pi\alpha/n\right]\cos\frac{2\pi s\alpha}{n}\right\} l_s.$$

Multiplying both sides of equation (A.λ) by $\frac{1}{n}e^{-\frac{2\pi i\lambda s}{n}}$ and summing for λ from 1 to n, we have, by using equations (3·3/s) and (3·4/s),

$$\ddot{k}_s - 2\omega\dot{l}_s = \left[3\omega^2+\frac{2m}{a^3}-\frac{m}{8a^3}\sum_{\alpha=1}^{n-1}\left(\frac{1}{\sin^3\frac{\pi\alpha}{n}}+\frac{1}{\sin\frac{\pi\alpha}{n}}\right)\right]k_s$$

$$+\frac{m}{a^3}\frac{1}{n}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\sum_{\mu}{}'\left[\frac{1+\sin^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n}+2\cos(\mu-\lambda)\,2\pi/n\right]\rho_\mu\right\}$$

$$+\frac{m}{a^3}\frac{1}{n}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\sum_{\mu}{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\pi/n}+1\right]\sin(\mu-\lambda)\,2\pi/n\,\sigma_\mu\right\}$$

$$=\left[3\omega^2+\frac{2m}{a^3}-\frac{m}{8a^3}\sum_{\alpha=1}^{n-1}\left(\frac{1}{\sin^3\frac{\pi\alpha}{n}}+\frac{1}{\sin\frac{\pi\alpha}{n}}\right)\right]k_s$$

$$+\frac{m}{a^3}k_s\left\{\sum_{\alpha=1}^{n-1}\left[\frac{1+\sin^2\frac{\pi\alpha}{n}}{8\sin^3\frac{\pi\alpha}{n}}+2\cos 2\pi\alpha/n\right]\cos 2\pi s\alpha/n\right\}$$

$$+i\frac{m}{a^3}\left\{\sum_{\alpha=1}^{n-1}\left[\frac{1}{16\sin^3\frac{\pi\alpha}{n}}+1\right]\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}\right\} l_s,$$

using the results proved above.

Multiplying both sides of equation (B.λ) by $\frac{1}{n}e^{-\frac{2\pi i\lambda s}{n}}$ and summing for λ from 1 to n, we have, by using equations ($3\cdot 3/s$) and ($3\cdot 4/s$),

$$\ddot{l}_s + 2\omega\dot{k}_s = \frac{m}{a^3}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\Sigma'_{\mu}\left[2 - \frac{1}{16\sin^3(\mu\sim\lambda)\pi/n}\right]\sin(\mu-\lambda)2\pi/n\,\rho_\mu\right\}$$

$$+\frac{m}{a^3}l_s\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\frac{\pi\alpha}{n}}{8\sin^3\frac{\pi\alpha}{n}} + \cos\frac{2\pi\alpha}{n}\right]$$

$$-\frac{m}{a^3}\times\frac{1}{n}\sum_{\lambda=1}^{n} e^{-\frac{2\pi i\lambda s}{n}}\left\{\Sigma'_{\mu}\left[\frac{1+\cos^2(\mu\sim\lambda)\pi/n}{8\sin^3(\mu\sim\lambda)\pi/n} + \cos(\mu-\lambda)2\pi/n\right]\sigma_\mu\right\}$$

$$= \frac{im}{a^3}\left\{\sum_{\alpha=1}^{n-1}\left[2 - \frac{1}{16\sin^3\frac{\pi\alpha}{n}}\right]\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}\right\}k$$

$$+\frac{m}{a^3}l_s\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\pi\alpha/n}{8\sin^3\pi\alpha/n} + \cos\frac{2\pi\alpha}{n}\right]$$

$$-\frac{m}{a^3}l_s\left\{\sum_{\alpha=1}^{n-1}\left[\frac{1+\cos^2\pi\alpha/n}{8\sin^3\pi\alpha/n} + \cos\frac{2\pi\alpha}{n}\right]\cos\frac{2\pi s\alpha}{n}\right\},$$

using the results stated above.

Let

$$\mathrm{N} = \frac{2m}{a^3} - \frac{m}{8a^3}\Sigma'_{\mu}\left\{\frac{1}{\sin^3(\mu\sim\lambda)\pi/n} + \frac{1}{\sin(\mu\sim\lambda)\pi/n}\right\}$$

$$= \frac{2m}{a^3} - \frac{m}{8a^3}\sum_{\alpha=}^{n-1}\left\{\frac{1}{\sin^3\frac{\pi\alpha}{n}} + \frac{1}{\sin\frac{\pi\alpha}{n}}\right\},$$

$$\mathrm{P}_s = \frac{m}{a^3}\Sigma'_{\mu}\left\{\frac{1}{8\sin^3(\mu\sim\lambda)\pi/n} + \frac{1}{8\sin(\mu\sim\lambda)\pi/n} + 2\cos(\mu-\lambda)2\pi/n\right\}\cos(\mu-\lambda)2\pi s/n$$

$$= \frac{m}{a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1}{8\sin^3\frac{\pi\alpha}{n}} + \frac{1}{8\sin\frac{\pi\alpha}{n}} + 2\cos 2\pi\alpha/n\right\}\cos 2\pi s\alpha/n,$$

$$\mathrm{Q}_s = \frac{m}{a^3}\Sigma'_{\mu}\left\{\frac{1}{16\sin^3(\mu\sim\lambda)\pi/n} + 1\right\}\sin(\mu-\lambda)2\pi/n\sin(\mu-\lambda)2\pi s/n$$

$$= \frac{m}{a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1}{16\sin^3\frac{\pi\alpha}{n}} + 1\right\}\sin 2\pi\alpha/n\sin 2\pi s\alpha/n,$$

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$$R_s = \frac{m}{a^3} \sum_{\mu}' \left\{ 2 - \frac{1}{16 \sin^3 (\mu \sim \lambda) \pi/n} \right\} \sin (\mu - \lambda) 2\pi/n \sin (\mu - \lambda) 2\pi s/n$$

$$= \frac{m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ 2 - \frac{1}{16 \sin^3 \frac{\pi\alpha}{n}} \right\} \sin 2\pi\alpha/n \sin 2\pi s\alpha/n,$$

$$T_s = \frac{2m}{a^3} \sum_{\mu}' \left\{ \frac{1 + \cos^2 (\mu \sim \lambda) \pi/n}{8 \sin^3 (\mu \sim \lambda) \pi/n} + \cos (\mu - \lambda) 2\pi/n \right\} \sin^2 (\mu - \lambda) \pi s/n$$

$$= \frac{2m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1 + \cos^2 \frac{\pi\alpha}{n}}{8 \sin^3 \frac{\pi\alpha}{n}} + \cos \frac{2\pi\alpha}{n} \right\} \sin^2 \frac{\pi s\alpha}{n}.$$

P_s, Q_s, R_s, and T_s depend on s only. The equations satisfied by k_s and l_s assume the form

$$\ddot{k}_s - 2\omega \dot{l}_s = [3\omega^2 + N + P_s] k_s + iQ_s l_s, \quad \ldots \ldots \quad (A'.s)$$

$$\ddot{l}_s + 2\omega \dot{k}_s = iR_s k_s + T_s l_s. \quad \ldots \ldots \quad (B'.s)$$

Thus we obtain n independent pairs of linear differential equations (A'.s) and (B'.s) where $s = 1, \ldots, n$, for the secondary dependent variables.

§ 4. *Comparison with Maxwell's Method*

It will be convenient, at this stage, to compare MAXWELL's method* of investigating the stability of the steady motion of a ring of particles of the same mass with that adopted in this paper.

MAXWELL assumed tacitly that the centre of Saturn could be taken as a Newtonian base, *i.e.*, as a " fixed " point, when the planet and the ring are assumed to form an " isolated " system with the law of gravitation as the law of force between any two members of the composite system. Now in Newtonian mechanics the centre of mass of a collection of particles forming an " isolated " system can be regarded as a Newtonian base, *i.e.*, a " fixed " point, for the motion of the system and not any one of the particles. The very significance of the term

$$\frac{m}{(r_\lambda^2 + z_\lambda^2)^{\frac{1}{2}}} - m \sum_{\mu}' \frac{r_\lambda r_\mu \cos (\theta_\lambda - \theta_\mu) + z_\lambda z_\mu}{(r_\mu^2 + z_\mu^2)^{\frac{3}{2}}}$$

in the general form of the force-function F_λ, defined in §1 of this paper, is that though the centre of the planet cannot be regarded as a Newtonian base for the general problem of the n particles of the ring and the planet, represented for mathematical purposes by a particle of mass M placed at O, with the law of gravitation

* *Loc. cit.*, Part II, §§ 1–7.

as the law of force, it has been rendered a base of reference for the motion of the particles of the ring by adding the above-mentioned term to

$$\frac{M}{(r_\lambda{}^2 + z_\lambda{}^2)^{\frac{1}{2}}} + m \sum_\mu{}' \frac{1}{\{r_\lambda{}^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu{}^2 + (z_\lambda - z_\mu)^2\}^{\frac{1}{2}}}.$$

This would have been the appropriate force-function for the motion of the particle λ relative to O, if O had been a Newtonian base for the motion of the particles of the ring with the law of gravitation as the law of force.

There is an interesting sequel to this statement. It is shown in § 8 of this paper that the steady motion is unconditionally unstable when $n = 2$, *i.e.*, if two particles of the same mass, at the opposite ends of a diameter, revolve with uniform angular velocity about O, the steady motion is unstable when the particles are disturbed in the plane of the circle. According to MAXWELL's method the steady motion could be stable provided the mass of the particles were sufficiently small. It is easy to see that the instability is solely due to the terms in the force-function F_λ which were ignored by MAXWELL.

Another point is also worth noticing. If we were to follow MAXWELL's method we should have had to write

$$\rho_\lambda = \sum_r A_r \begin{matrix}\sin \\ \cos\end{matrix}\left(p_r t + \frac{2\pi\lambda r}{n} + \gamma_r\right), \qquad \sigma_\lambda = \sum_r B_r \begin{matrix}\cos \\ \sin\end{matrix}\left(p_r t + \frac{2\pi\lambda r}{n} + \gamma_r\right),$$

where p_r, A_r, B_r, and γ_r are constants. Then following MAXWELL* p_r would have to satisfy an equation similar to, but not identical with, the "stability" equation (C.s), arrived at in § 6 of this paper. If we assume that the number of terms in the summation is finite, we prescribe arbitrarily the form of the solutions of the differential equations. On the other hand, if, following MAXWELL, we regard the above formulæ as "FOURIER expansions" for ρ_λ and σ_λ with $\frac{2\pi\lambda}{n} = x$ as the argument, we must take cognizance of the fact that ρ_λ and σ_λ are defined only for a discrete set of values of x, viz., for $\frac{2\pi\lambda}{n}$ $(\lambda = 1, \ldots, n)$.

The writer has verified that by employing formulæ $(3\cdot1/\lambda)$ and $(3\cdot2/\lambda)$ and by using solutions of the equations (A'.s) and (B'.s) one can obtain formulæ for ρ_λ and σ_λ which are real functions of t containing $4n$ real arbitrary constants as required by the theory of differential equations. The formulæ thus obtained are practically of the form stated above.

The model of a ring adopted in §1 of this paper is precisely the same as that of MAXWELL. It was the inapplicability of the GOLDSBROUGH hypothesis, discussed in §13, which forms Part II of this paper, which led the writer to the transformation formulæ introduced in § 3.

* *Loc. cit.*, Part II, § 6, equation (22) for "n."

§ 5. *The case $s = n$: formulæ for k_n and l_n*

The equations (A′.s) and (B′.s) become, when $s = n$,

$$\ddot{k}_n - 2\omega \dot{l}_n = 3\omega^2 k_n$$

$$\ddot{l}_n + 2\omega \dot{k}_n = 0.$$

The solutions of these equations are

$$\left.\begin{aligned} k_n &= \frac{2\,d_n}{\omega} + a_n e^{i\omega t} + b_n e^{-i\omega t} \\ l_n &= c_n - 3\,d_n t + 2i\,(a_n e^{i\omega t} - b_n e^{-i\omega t}) \end{aligned}\right\} \begin{array}{l} a_n,\ b_n,\ c_n,\ d_n \text{ being} \\ \text{arbitrary constants.} \end{array}$$

The secular term does not denote instability, for the coefficient of l_n is the same in all the equations of transformation.

§ 6. "*Stability*" *Equations*

The equations (A′.n) and (B′.n) were the simplest of the set (A′) and (B′). Considering a general pair (A′.s) and (B′.s) we assume, according to the usual method, $k_s = a_s e^{ip_s t}$, $l_s = b_s e^{ip_s t}$. Then we have,

$$a_s\,[(3\omega^2 + \mathrm{N} + \mathrm{P}_s) + p_s^2] = -\,ib_s\,[2\omega p_s + \mathrm{Q}_s]$$

$$b_s\,[\mathrm{T}_s + p_s^2] = ia_s\,[2\omega p_s - \mathrm{R}_s].$$

Eliminating a_s and b_s,

$$(p_s^2 + \mathrm{T}_s)\,[p_s^2 + (3\omega^2 + \mathrm{N} + \mathrm{P}_s)] = (2\omega p_s + \mathrm{Q}_s)\,(2\omega p_s - \mathrm{R}_s).$$

i.e.,

$$p_s^4 + (-\,\omega^2 + \mathrm{N} + \mathrm{P}_s + \mathrm{T}_s)\,p_s^2 + 2\omega\,(\mathrm{R}_s - \mathrm{Q}_s)\,p_s + \{\mathrm{T}_s\,(3\omega^2 + \mathrm{N} + \mathrm{P}_s) + \mathrm{Q}_s\mathrm{R}_s\} = 0. \quad . \; . \; . \; . \; . \; . \; . \quad (\mathrm{C}.s)$$

We obtain n equations, of which (C.s) is a typical member. The condition for stability is that the roots of every one of the set (C.s) be real.

§ 7. *Maxwell's Method**

In §§ 7–10 the nature of the roots of the equations (C.s) is investigated. In § 7 MAXWELL's method is described. In § 8 it is shown that the equation (C.1) has two imaginary roots when $n = 2$; *i.e.*, the steady motion is unstable when $n = 2$. It also appears that the equation (C.1) would have two complex roots when

* *Loc. cit.*, Part II, § 7.

$n = 3, 4, 5, 6$ (§10). In § 9 it is shown that the roots of the equation (C.s), where $1 < s < n - 1$ and $n > 2$, will be real when M is sufficiently large.

It is convenient to represent MAXWELL's method graphically.

Putting

$$U \equiv p^4 + (-\omega^2 + N + P_s + T_s)\, p^2 + 2\omega\,(R_s - Q_s)\, p + \{T_s\,(3\omega^2 + N + P_s) + Q_s R_s\}$$

the graph of U against p is plotted. Then, following MAXWELL, provided M and ω^2 are large and $T_s > 0$, the graph is represented by fig. 1.

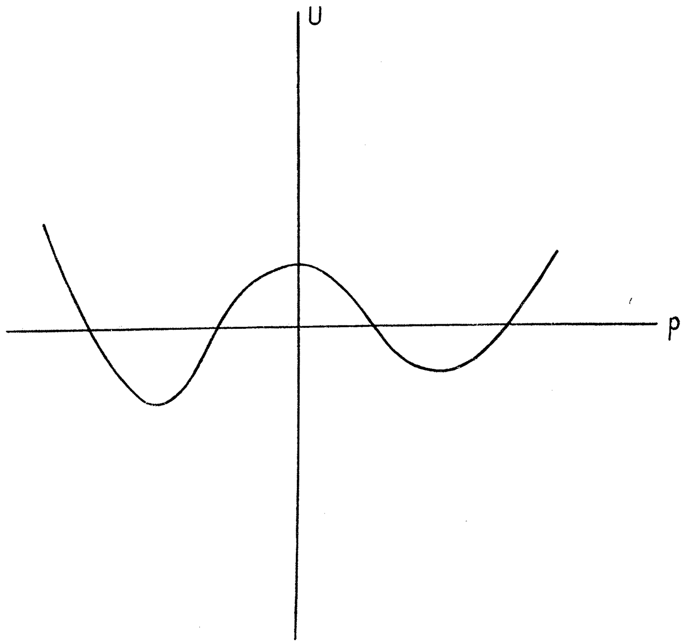

FIG. 1

MAXWELL concluded that the roots of every equation of the type (C.s) were real provided M and ω^2 were sufficiently large.

For a thorough theoretical consideration of the equations (C.s) it is necessary to know the exact values of the basic quantities

$$\sum_{\alpha=1}^{n-1} \frac{1}{\sin \frac{\pi\alpha}{n}} \quad \text{and} \quad \sum_{\alpha=1}^{n-1} \frac{\cos^2 \frac{\pi\alpha}{n}}{\sin^3 \frac{\pi\alpha}{n}};$$

the quantities P_s, etc., can, in the general case, be evolved from them. The basic quantities

$$\sum_{\alpha=1}^{n-1} \frac{1}{\sin \frac{\pi\alpha}{n}} \quad \text{and} \quad \sum_{\alpha=1}^{n-1} \frac{\cos^2 \frac{\pi\alpha}{n}}{\sin^3 \frac{\pi\alpha}{n}}$$

can only be evaluated if the value of

$$\prod_{\alpha=1}^{n-1} \tan\left(\theta + \frac{\alpha\pi}{2n}\right)$$

is known in terms of θ and n. Unfortunately, it has not been possible to evaluate the product ; consequently a full examination of the equations is not possible.

§ 8. *Case* $n = 2$

The case $s = n$ has been considered before, and we shall consider the case $n = 2$, $s = 1$.

$$\omega^2 a^3 = \mathrm{M} + \tfrac{1}{4}m,$$

$$\mathrm{N} = \tfrac{7}{4}\frac{m}{a^3},$$

$$\mathrm{Q}_1 = 0,\ \mathrm{R}_1 = 0,\ \mathrm{T}_1 = -\tfrac{7}{4}\frac{m}{a^3}, \qquad \mathrm{P}_1 = \tfrac{7}{4}\frac{m}{a^3},$$

$$\mathrm{T}_1(3\omega^2 + \mathrm{N} + \mathrm{P}_1) + \mathrm{Q}_1\mathrm{R}_1 = -\tfrac{7}{4}\frac{m}{a^3}(3\omega^2 + \mathrm{N} + \mathrm{P}_1) + 0$$

$$= -\tfrac{7}{4}\frac{m}{a^3}\left(\frac{3\mathrm{M}}{a^3} + \tfrac{17}{4}\frac{m}{a^3}\right) = -\tfrac{7}{4}\frac{m}{a^6}(3\mathrm{M} + \tfrac{17}{4}m),$$

$$-\omega^2 + \mathrm{N} + \mathrm{P}_1 + \mathrm{T}_1 = -\frac{1}{a^3}(\mathrm{M} + \tfrac{1}{4}m) + \tfrac{7}{4}\frac{m}{a^3} = -\frac{1}{a^3}(\mathrm{M} - \tfrac{3}{2}m),$$

therefore, the equation (C.1) becomes

$$p_1^4 - \frac{1}{a^3}(\mathrm{M} - \tfrac{3}{2}m)\,p_1^2 - \tfrac{7}{4}\frac{m}{a^6}(3\mathrm{M} + \tfrac{17}{4}m) = 0.$$

The independent term is negative for all positive values of M and m, which, of course, they have.

Therefore, p_1^2 will have one negative value and the equation (C.1) will have two imaginary roots which are

$$\pm\sqrt{\frac{1}{2a^3}\{(\mathrm{M} - \tfrac{3}{2}m) - \sqrt{(\mathrm{M} - \tfrac{3}{2}m)^2 + 7m(3\mathrm{M} + \tfrac{17}{4}m)}\}}.$$

Hence, according to our method of reckoning, the steady motion will be unstable. The investigation has been carried to quantities of the second order, and the instability has been confirmed.

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§ 9. *The case* $n > 2$ *and* $s \neq 1$ *or* $(n-1)$ or n

For $s \neq 1$, $n-1$ we have

$$\sum_{\alpha=1}^{n-1} 2\cos\frac{2\pi\alpha}{n}\cos\frac{2\pi s\alpha}{n} = -2,$$

$$\sum_{\alpha=1}^{n-1} \sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n} = 0,$$

and

$$\sum_{\alpha=1}^{n-1} \cos\frac{2\pi\alpha}{n}\sin^2\frac{\pi s\alpha}{n} = 0.$$

Then we should have

$$N + P_s = -\frac{m}{4a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1}{\sin^3\frac{\pi\alpha}{n}} + \frac{1}{\sin\frac{\pi\alpha}{n}}\right\}\sin^2\frac{\pi s\alpha}{n}$$

$$Q_s = \frac{m}{16a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}}{\sin^3\frac{\pi\alpha}{n}}\right\}$$

$$R_s = -\frac{m}{16a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{\sin\frac{2\pi\alpha}{n}\sin\frac{2\pi s\alpha}{n}}{\sin^3\frac{\pi\alpha}{n}}\right\}$$

$$T_s = \frac{m}{4a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1+\cos^2\frac{\pi\alpha}{n}}{\sin^3\frac{\pi\alpha}{n}}\sin^2\frac{\pi s\alpha}{n}\right\}.$$

$T_s > 0$ for all values of s other than 1 or $(n-1)$ or n.

Since $T_s > 0$ when $s \neq 1, \overline{n-1}, n$, MAXWELL's method and the graphical representation will apply to all the equations (C.s) except for $s = 1, n-1, n$.

§ 10. *The case* $n > 2$, $s = 1$

We have, carrying out all the summations,

$$N + P_1 = \frac{m}{a^3}\left[n - \tfrac{1}{4}\cot\frac{\pi}{2n} - \tfrac{1}{4}\sum_{\alpha=1}^{n-1}\frac{1}{\sin\frac{\pi\alpha}{n}}\right]$$

$$T_1 = \frac{m}{2a^3}\left[\sum_{\alpha=1}^{n-1}\frac{1}{\sin\frac{\pi\alpha}{n}} - \left(n + \tfrac{1}{2}\cot\frac{\pi}{2n}\right)\right]$$

$$Q_1 = \frac{m}{2a^3}\left[\tfrac{1}{2}\sum_{\alpha=1}^{n-1}\frac{1}{\sin\frac{\pi\alpha}{n}} + n - \tfrac{1}{2}\cot\frac{\pi}{2n}\right]$$

$$R_1 = \frac{m}{a^3}\left[n + \tfrac{1}{4}\cot\frac{\pi}{2n} - \tfrac{1}{4}\sum_{\alpha=1}^{n-1}\frac{1}{\sin\dfrac{\pi\alpha}{n}}\right]$$

$$R_1 - Q_1 = \frac{m}{2a^3}\left[\left(n + \cot\frac{\pi}{2n}\right) - \sum_{\alpha=1}^{n-1}\frac{1}{\sin\dfrac{\pi\alpha}{n}}\right]$$

$$-\omega^2 + N + P_1 + P_1 = -\frac{1}{a^3}\left(M + \tfrac{1}{2}m\cot\frac{\pi}{2n} - \tfrac{1}{2}mn\right)$$

$$3\omega^2 + N + P_1 = \frac{1}{a^3}\left[3M + \tfrac{1}{2}m\sum_{\alpha=1}^{n-1}\frac{1}{\sin\dfrac{\pi\alpha}{n}} + mn - \tfrac{1}{4}m\cot\frac{\pi}{2n}\right]$$

$$T_1(3\omega^2 + N + P_1) + Q_1R_1 = 3\left(\omega^2 - \frac{M}{a^3}\right)\left(\omega^2 + \frac{M}{a^3}\right)$$
$$-\frac{m}{2a^3}\left[\omega^2\left(2\cot\frac{\pi}{2n} - 3n\right) + \frac{M}{a^3}\left(6n - \tfrac{5}{2}\cot\frac{\pi}{2n}\right)\right] - \frac{1}{2}\frac{m^2}{a^6}\,n\cot\frac{\pi}{2n}.$$

The graph in fig. 1 will be applicable to the equation (C.1) for large values of M, only if $T_1 > 0$. If $T_1 < 0$, the graph will be represented by fig. 2.

No convenient formula for T_1 in terms of n is known.

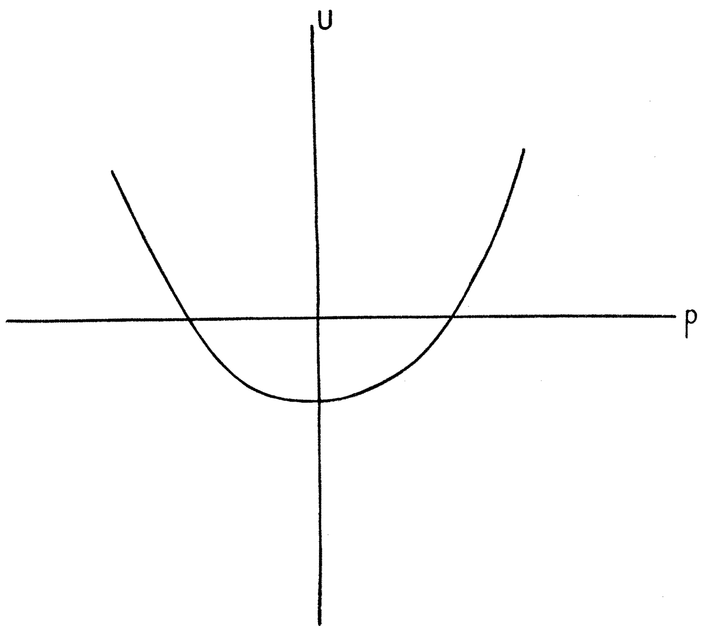

FIG. 2

If we put

$$f(n) = \sum_{\alpha=1}^{n-1}\frac{1}{\sin\dfrac{\pi\alpha}{n}}, \quad \phi(n) = f(n) - \left(n + \tfrac{1}{2}\cot\frac{\pi}{2n}\right),$$

we can write

$$T_1 = \frac{m}{2a^3}\,\phi\,(n).$$

We obtain the following table of values :—

n	$f(n)$	$\phi(n)$
3	2·308	− 1·858
4	3·828	− 1·3791
5	5·5055	− 1·0333
6	7·308	− 0·558
7	9·2098	0·0197
9	13·2995	1·4160
12	.	4·138

$T_1 < 0$ for $n = 3, 4, 5, 6$. It appears that after $n > 6$, T_1 would be positive. Hence the graphs for $n = 3, 4, 5, 6$ for the equations (C.1) and (C. $n - 1$) will be of the type represented by fig. 2 for large values of M. Each of the two equations (C.1) and (C. $\overline{n - 1}$) will have two complex roots and the motion will not be stable.

We cannot make any definite remarks about the roots when $n > 7$. It is likely that they would be real.

§11. Maxwell's *limit for stability* (*for large values of n*)—Maxwell assumed that there was one equation among his equations, similar to the " stability " equations (C.s), of this paper (§6), the reality of the roots of which ensured the reality of the roots of the other equations. He found the criterion which would have to be satisfied in order that the roots of this fundamental " stability " equation be real. This criterion is investigated here.

According to the method followed in this paper, Maxwell's method can be applied only when n is even.

Let us suppose that n is even. Then (C·$\frac{1}{2}n$) is the equation which was regarded as fundamental by Maxwell. We proceed to find the condition that the roots of *this* equation be all real. The criterion will have an asymptotic nature.

$Q_{\frac{1}{2}n} = 0$, $R_{\frac{1}{2}n} = 0$, and we shall have a quadratic equation for $p_{\frac{1}{2}n}{}^2$. Put $n = 2\nu$ where ν is a positive integer.

We shall suppose that n is large and will, therefore, find the asymptotic values of the quantities P_ν, T_ν, and N.

Since $s = \frac{1}{2}n = \nu$ is different from 1 or $(n - 1)$, we have

$$N + P_\nu = -\frac{m}{4a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1}{\sin^3\frac{\alpha\pi}{n}} + \frac{1}{\sin\frac{\alpha\pi}{n}}\right\}\sin^2\tfrac{1}{2}\alpha\pi,$$

$$T_\nu = \frac{m}{4a^3}\sum_{\alpha=1}^{n-1}\left\{\frac{1+\cos^2\frac{\alpha\pi}{n}}{\sin^3\frac{\alpha\pi}{n}}\sin^2\tfrac{1}{2}\alpha\pi\right\}.$$

Remembering that $n = 2\nu$, we have

$$N + P_\nu = -\frac{m}{4a^3}\sum_{\alpha=1}^{2\nu-1}\left\{\frac{2}{\sin\frac{\pi\alpha}{n}} + \frac{\cos^2\frac{\alpha\pi}{n}}{\sin^3\frac{\alpha\pi}{n}}\right\}\sin^2\tfrac{1}{2}\alpha\pi,$$

and

$$T_\nu = \frac{m}{4a^3}\sum_{\alpha=1}^{2\nu-1}\left\{\frac{1}{\sin\frac{\pi\alpha}{n}} + \frac{2\cos^2\frac{\pi\alpha}{n}}{\sin^3\frac{\pi\alpha}{n}}\right\}\sin^2\tfrac{1}{2}\alpha\pi.$$

So we have to evaluate the quantities

$$\sum_{\alpha=1}^{2\nu-1}\frac{\sin^2\frac{1}{2}\alpha\pi}{\sin\frac{\alpha\pi}{n}} \quad\text{and}\quad \sum_{\alpha=1}^{2\nu-1}\frac{\cos^2\frac{\alpha\pi}{n}\sin^2\frac{1}{2}\alpha\pi}{\sin^3\frac{\alpha\pi}{n}}.$$

We shall use RIEMANN's definition of an integral as the limit of a sum. The values thus obtained will be rough estimates.

$$\sum_{\alpha=1}^{2\nu-1}\frac{\sin^2\frac{1}{2}\alpha\pi}{\sin\frac{\alpha\pi}{n}} = \frac{1}{\sin\frac{\pi}{2\nu}} + \frac{1}{\sin\frac{3\pi}{2\nu}} + \dots + \frac{1}{\sin\frac{(2\nu-1)\pi}{2\nu}}$$

$$= \frac{1}{\sin\left(\frac{\pi}{\nu}-\frac{\pi}{2\nu}\right)} + \frac{\nu}{\pi}\left[\frac{1}{\sin\left(\frac{2\pi}{\nu}-\frac{\pi}{2\nu}\right)}\frac{\pi}{\nu} + \frac{1}{\sin\left(\frac{3\pi}{\nu}-\frac{\pi}{2\nu}\right)}\frac{\pi}{\nu} + \dots + \frac{1}{\sin\left(\frac{\nu\pi}{\nu}-\frac{\pi}{2\nu}\right)}\frac{\pi}{\nu}\right]$$

$$\simeq \frac{1}{\sin\frac{\pi}{2\nu}} + \frac{\nu}{\pi}\int_{\pi/\nu}^{\pi}\frac{dx}{\sin\left(x-\frac{\pi}{2\nu}\right)}$$

$$= \frac{1}{\sin\frac{\pi}{2\nu}} + \frac{\nu}{\pi}\log\cot^2\frac{\pi}{4\nu}$$

$$= \frac{1}{\sin\frac{\pi}{n}} + \frac{n}{\pi}\log\cot\frac{\pi}{2n}, \text{ using the relation } n = 2\nu$$

$$\simeq \frac{n}{\pi} + \frac{n}{\pi}\log\left(\frac{2n}{\pi}\right).$$

By exactly similar reasoning

$$\sum_{\alpha=1}^{2\nu-1} \frac{\cos^2 \frac{\alpha\pi}{n}}{\sin^3 \frac{\alpha\pi}{n}} \sin^2 \tfrac{1}{2}\alpha\pi \simeq \frac{\cos^2 \frac{\pi}{2\nu}}{\sin^3 \frac{\pi}{2\nu}} + \frac{\nu}{\pi} \int_{\pi/\nu}^{\pi} \frac{\cos^2 \left(x - \frac{\pi}{2\nu}\right)}{\sin^3 \left(x - \frac{\pi}{2\nu}\right)} dx$$

$$\int_{\pi/\nu}^{\pi} \frac{\cos^2 \left(x - \frac{\pi}{2\nu}\right)}{\sin^3 \left(x - \frac{\pi}{2\nu}\right)} dx = \left[-\tfrac{1}{2} \frac{\cos \left(x - \frac{\pi}{2\nu}\right)}{\sin^2 \left(x - \frac{\pi}{2\nu}\right)} \right]_{\pi/\nu}^{\pi} - \tfrac{1}{2} \int_{\pi/\nu}^{\pi} \frac{dx}{\sin \left(x - \frac{\pi}{2\nu}\right)}$$

$$= \frac{\cos \frac{\pi}{n}}{\sin^2 \frac{\pi}{n}} - \log \cot \frac{\pi}{2n}$$

$$\simeq \frac{n^2}{\pi^2} - \log \left(\frac{2n}{\pi}\right).$$

Therefore, we have the asymptotic formulæ

$$\sum_{\alpha=1}^{2\nu-1} \frac{\sin^2 \frac{1}{2}\alpha\pi}{\sin \frac{\alpha\pi}{n}} \simeq \frac{n}{\pi} + \frac{n}{\pi} \log \frac{2n}{\pi} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11.1)$$

$$\sum_{\alpha=1}^{2\nu-1} \frac{\cos^2 \frac{\alpha\pi}{n}}{\sin^3 \frac{\alpha\pi}{n}} \sin^2 \tfrac{1}{2}\alpha\pi \simeq \tfrac{3}{2} \frac{n^3}{\pi^3} - \tfrac{1}{2} \frac{n}{\pi} \log \frac{2n}{\pi} \quad . \; . \; . \; . \; . \quad (11.2)$$

for large values of n.

Hence, if we decide to retain only terms of the order of n^3, we shall have

$$N + P_\nu \simeq -\tfrac{3}{8} \frac{m}{a^3} \frac{n^3}{\pi^3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11.3)$$

$$T_\nu \simeq \tfrac{3}{4} \frac{m}{a^3} \frac{n^3}{\pi^3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11.4)$$

for large values of n.

The equation (C.ν) then becomes

$$p_\nu^4 + \left(-\omega^2 + \tfrac{3}{8} \frac{m}{a^3} \frac{n^3}{\pi^3}\right) p_\nu^2 + \tfrac{3}{4} \frac{m}{a^3} \frac{n^3}{\pi^3} \left(3\omega^2 - \tfrac{3}{8} \frac{m}{a^3} \frac{n^3}{\pi^3}\right) = 0\,;$$

or, as far as quantities of the order of n^3 are concerned,

$$p_\nu^4 - \frac{1}{a^3} \left(M - \tfrac{3}{8} \frac{n^3}{\pi^3} m\right) p_\nu^2 + \tfrac{9}{4} \frac{1}{a^6} m \left(M - \tfrac{1}{8} \frac{n^3}{\pi^3} m\right) \frac{n^3}{\pi^3} = 0. \quad . \; . \; . \; . \; . \; . \quad (11.5)$$

The roots of the equation (11.5) must be real. The conditions are

$$M > \tfrac{3}{8}\frac{n^3}{\pi^3}m,$$

$$M > \tfrac{1}{8}\frac{n^3}{\pi^3}m,$$

and

$$\left(M - \tfrac{3}{8}\frac{n^3}{\pi^3}m\right)^2 > 9m\left(M - \tfrac{1}{8}\frac{n^3}{\pi^3}m\right)\frac{n^3}{\pi^3},$$

since both the values of p_ν^2 as given by (11.5) must be positive.

The last condition becomes

$$M^2 - \tfrac{3}{4}mM\frac{n^3}{\pi^3} + \tfrac{9}{64}\frac{n^6}{\pi^6}m^2 > 9\frac{n^3}{\pi^3}mM - \tfrac{9}{8}\frac{n^6}{\pi^6}m^2$$

i.e.,

$$M^2 - \tfrac{39}{4}\frac{n^3}{\pi^3}mM + \tfrac{81}{64}\frac{n^6}{\pi^6}m^2 > 0,$$

i.e.,

$$\left\{M - \tfrac{39}{8}\frac{n^3}{\pi^3}m - \frac{n^3}{2\pi^3}m\sqrt{(\tfrac{39}{4})^2 - \tfrac{81}{16}}\right\}\left\{M - \tfrac{39}{8}\frac{n^3}{\pi^3}m + \frac{n^3}{2\pi^3}m\sqrt{(\tfrac{39}{4})^2 - \tfrac{81}{16}}\right\} > 0$$

i.e.,

$$\text{either } M > 38\cdot47\frac{mn^3}{4\pi^3} \text{ or } M < 0\cdot53\frac{mn^3}{4\pi^3}.$$

Hence the conditions which have to be satisfied in order that the equation (11.5) may have real roots are

(i) $M > \tfrac{3}{8}\dfrac{mn^3}{\pi^3}$,

(ii) $M > \tfrac{1}{8}\dfrac{mn^3}{\pi^3}$,

(iii) either $M > 38\cdot47\dfrac{mn^3}{4\pi^3}$ or $M < 0\cdot53\dfrac{mn^3}{4\pi^3}$.

(i) is incompatible with the condition $M < 0\cdot53\dfrac{mn^3}{4\pi^3}$.

Hence the condition that has to be satisfied in order that the equation (11.5) may have real roots is

$$M > 38\cdot47\frac{mn^3}{4\pi^3},\ i.e.,\ mn^3 < \frac{4\pi^3}{38\cdot47}M,$$

or

$$M > 0\cdot3102\,mn^3,\ i.e.,\ mn^3 < 3\cdot223\,M. \quad \ldots\ldots \quad (11.6)$$

The original MAXWELL criterion* is

$$M > 0\cdot4352\,mn^3,\ i.e.,\ mn^3 < 2\cdot298\,M. \quad \ldots\ldots \quad (11.7)$$

* *Loc. cit.*, Part II, § 8, pp. 318–319.

§12. *The connection between the motion of the centre of mass of the system of particles and the secondary dependent variables.*

Let $(\xi, \eta, 0)$ be the co-ordinates of G, the centre of mass of the system, relative to the frame Oxyz. Let R $= mn$ be the total mass of the system of particles.

Let $\chi = \omega t + \varepsilon$, so that for the particle λ we have $\theta_\lambda = \chi + \frac{2\pi\lambda}{n} + \sigma_\lambda$ in the disturbed motion. By well-known formulæ we have

$$\xi = \frac{a}{n} \sum_{\lambda=1}^{n} (1 + \rho_\lambda) \cos\left(\chi + \frac{2\pi\lambda}{n} + \sigma_\lambda\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.1)$$

$$\eta = \frac{a}{n} \sum_{\lambda=1}^{n} (1 + \rho_\lambda) \sin\left(\chi + \frac{2\pi\lambda}{n} + \sigma_\lambda\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.2)$$

By the principles of mechanics we have, for the motion of G,

$$\ddot{\xi} = -\frac{(\mathrm{M} + \mathrm{R})}{a^2 n} \sum_{\lambda=1}^{n} \frac{\cos\left(\chi + \frac{2\pi\lambda}{n} + \sigma_\lambda\right)}{(1 + \rho_\lambda)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.3)$$

$$\ddot{\eta} = -\frac{(\mathrm{M} + \mathrm{R})}{a^2 n} \sum_{\lambda=1}^{n} \frac{\sin\left(\chi + \frac{2\pi\lambda}{n} + \sigma_\lambda\right)}{(1 + \rho_\lambda)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.4)$$

So far the equations are exact. Retaining now terms of the first order we obtain the following approximate results :—

$$\xi = \tfrac{1}{2} a \left[(k_{n-1} e^{i\chi} + k_1 e^{-i\chi}) + i (l_{n-1} e^{i\chi} - l_1 e^{-i\chi})\right] \quad . \quad . \quad . \quad . \quad . \quad (12.5)$$

$$\eta = \tfrac{1}{2} a \left[(l_{n-1} e^{i\chi} + l_1 e^{-i\chi}) - i (k_{n-1} e^{i\chi} - k_1 e^{-i\chi})\right] \quad . \quad . \quad . \quad . \quad . \quad (12.6)$$

$$\ddot{\xi} = \frac{2(\mathrm{M} + \mathrm{R})}{a^3} \xi - \frac{3(\mathrm{M} + \mathrm{R}) i}{2a^2} (l_{n-1} e^{i\chi} - l_1 e^{-i\chi}) \quad . \quad . \quad . \quad . \quad (12.7)$$

$$\ddot{\eta} = \frac{2(\mathrm{M} + \mathrm{R})}{a^3} \eta - \frac{3(\mathrm{M} + \mathrm{R})}{2a^2} (l_{n-1} e^{i\chi} + l_1 e^{-i\chi}). \quad . \quad . \quad . \quad . \quad . \quad (12.8)$$

Hence we see that ξ and η are connected explicitly with k_1, k_{n-1}, l_1, and l_{n-1} only.

PART II

Perturbations due to a Satellite

§13. *The equations of the perturbed motion of the system of particles moving in the datum state of* § 2 *about* O *due to a satellite moving relative to* O *in a circle in the plane of the ring.*

It is assumed that the effect of the system of particles on the motion of the satellite relative to O can be ignored.

Let

m' = mass of the satellite.

a' = the radius of the circle in which the satellite moves about O.

ω' = angular velocity of the satellite in its orbit, and let $(r', \theta', 0)$ be the cylindrical polar co-ordinates of the satellite relative to the frame Oxyz.

Then $r' = a'$, $\theta' = \omega' t + \varepsilon'$, where ε' is an arbitrary constant.

The problem is clearly two-dimensional.

In the datum state $(r_\lambda)_0 = a$, $(\theta_\lambda)_0 = \omega t + \varepsilon + \dfrac{2\pi\lambda}{n}$ where ε is an arbitrary constant.

In the perturbed state, let

$r_\lambda = a(1 + \rho_\lambda)$, $\theta_\lambda = \omega t + \varepsilon + \dfrac{2\pi\lambda}{n} + \sigma_\lambda$, where the ρ_λ's, the σ_λ's, and their derivatives are assumed to be so small that their products may be neglected.

As in § 2 we take

$$F_\lambda = \frac{(M + m)}{r_\lambda} + \Sigma'_{\mu} m \left[\frac{1}{\{r_\lambda^2 - 2 r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + r_\mu^2\}^{\frac{1}{2}}} - \frac{r_\lambda \cos(\theta_\lambda - \theta_\mu)}{r_\mu^2}\right]$$

and

$$\Omega_\lambda = \frac{m'}{\{r'^2 + r_\lambda^2 - 2 r' r_\lambda \cos(\theta' - \theta_\lambda)\}^{\frac{1}{2}}} - \frac{m' r_\lambda \cos(\theta' - \theta_\lambda)}{r'^2}$$

$$= \frac{m'}{\{a'^2 - 2 a' r_\lambda \cos(\theta' - \theta_\lambda) + r_\lambda^2\}^{\frac{1}{2}}} - \frac{m' r_\lambda \cos(\theta' - \theta_\lambda)}{a'^2}.$$

Let

$$\alpha = \frac{a}{a'} \quad \text{and} \quad \phi_\lambda = (\omega' - \omega)\, t + \varepsilon' - \varepsilon - \frac{2\pi\lambda}{n};$$

we assume that

$$\frac{1}{(1 - 2\alpha\cos\phi_\lambda + \alpha^2)^{\frac{1}{2}}} = \tfrac{1}{2} b_0 + b_1 \cos\phi_\lambda + b_2 \cos 2\phi_\lambda + \ldots\ldots,$$

the b's being functions of α.

It is also assumed that the series for $(1 - 2\alpha\cos\phi_\lambda + \alpha^2)^{-\frac{1}{2}}$ can be differentiated term by term with respect to α and ϕ_λ. α is positive and less than one.*

The equations of the perturbed motion of the particle λ are, as in §1,

$$a\{\ddot\rho_\lambda - 2\omega\dot\sigma_\lambda - \omega^2\rho_\lambda - \omega^2\} = \left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left(\frac{\partial^2 F_\lambda}{\partial r_\mu \partial r_\lambda}\right)_0 + \sum_\mu \sigma_\mu \left(\frac{\partial^2 F_\lambda}{\partial\theta_\mu \partial r_\lambda}\right)_0$$
$$+ \left(\frac{\partial\Omega_\lambda}{\partial r_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left(\frac{\partial^2\Omega_\lambda}{\partial r_\mu \partial r_\lambda}\right)_0 + \sum_\mu \sigma_\mu \left(\frac{\partial^2\Omega_\lambda}{\partial\theta_\mu \partial r_\lambda}\right)_0,$$

$$a\{\ddot\sigma_\lambda + 2\omega\dot\rho_\lambda\} = \left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial\theta_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left\{\frac{\partial}{\partial r_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial\theta_\lambda}\right)\right\}_0 + \sum_\mu \sigma_\mu \left\{\frac{\partial}{\partial\theta_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial\theta_\lambda}\right)\right\}_0$$
$$+ \left(\frac{1}{r_\lambda}\frac{\partial\Omega_\lambda}{\partial\theta_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left\{\frac{\partial}{\partial r_\mu}\left(\frac{1}{r_\lambda}\frac{\partial\Omega_\lambda}{\partial\theta_\lambda}\right)\right\}_0 + \sum_\mu \sigma_\mu \left\{\frac{\partial}{\partial\theta_\mu}\left(\frac{1}{r_\lambda}\frac{\partial\Omega_\lambda}{\partial\theta_\lambda}\right)\right\}_0.$$

* The symbol "α" used here should not be confused with the variable suffix "α" ranging from 1 to $n - 1$.

The values of

$$\sum_\mu a\rho_\mu \left(\frac{\partial^2 F_\lambda}{\partial r_\mu \partial r_\lambda}\right)_0 + \sum_\mu \sigma_\mu \left(\frac{\partial^2 F_\lambda}{\partial \theta_\mu \partial r_\lambda}\right)_0$$

and

$$\sum_\mu a\rho_\mu \left\{\frac{\partial}{\partial r_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)\right\}_0 + \sum_\mu \sigma_\mu \left\{\frac{\partial}{\partial \theta_\mu}\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)\right\}_0,$$

can be taken over from §2. We have to find the values of the other quantities on the right-hand sides of the above equations.

As far as the r_λ's and the θ_λ's are concerned, Ω_λ is a function of r_λ and θ_λ alone. Therefore

$$\sum_\mu a\rho_\mu \left\{\frac{\partial^2 \Omega_\lambda}{\partial r_\mu \partial r_\lambda}\right\}_0 + \sum_\mu \sigma_\mu \left\{\frac{\partial^2 \Omega_\lambda}{\partial \theta_\mu \partial r_\lambda}\right\}_0 = a\rho_\lambda \left(\frac{\partial^2 \Omega_\lambda}{\partial r_\lambda^2}\right)_0 + \sigma_\lambda \left(\frac{\partial^2 \Omega_\lambda}{\partial \theta_\lambda \partial r_\lambda}\right)_0,$$

and

$$\sum_\mu a\rho_\mu \left\{\frac{\partial}{\partial r_\mu}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0 + \sum_\mu \sigma_\mu \left\{\frac{\partial}{\partial \theta_\mu}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0 = a\rho_\lambda \left\{\frac{\partial}{\partial r_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0 + \sigma_\lambda \left\{\frac{\partial}{\partial \theta_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0.$$

$$\Omega_\lambda = \frac{m'}{\{a'^2 - 2a' r_\lambda \cos(\theta' - \theta_\lambda) + r_\lambda^2\}^{\frac{1}{2}}} - \frac{m' r_\lambda \cos(\theta' - \theta_\lambda)}{a'^2}$$

$$= \frac{m'}{a'} \frac{1}{\left\{1 - 2\frac{r_\lambda}{a'}\cos(\theta' - \theta_\lambda) + \frac{r_\lambda^2}{a'^2}\right\}^{\frac{1}{2}}} - \frac{m'}{a'^2} r_\lambda \cos(\theta' - \theta_\lambda).$$

Carrying out all the differentiations we obtain*

$$\left(\frac{\partial \Omega_\lambda}{\partial r_\lambda}\right)_0 + a\rho_\lambda \left(\frac{\partial^2 \Omega_\lambda}{\partial r_\lambda^2}\right)_0 + \sigma_\lambda \left(\frac{\partial^2 \Omega_\lambda}{\partial \theta_\lambda \partial r_\lambda}\right)_0$$

$$= \frac{m'}{a'^2}\frac{\partial}{\partial \alpha}(\tfrac{1}{2}b_0 + b_1 \cos\phi_\lambda + \ldots + b_q \cos q\phi_\lambda + \ldots) - \frac{m'}{a'^2}\cos\phi_\lambda$$

$$+ \frac{m'}{a'^3}\frac{\partial^2}{\partial \alpha^2}\{\tfrac{1}{2}b_0 + b_1 \cos\phi_\lambda + \ldots + b_q \cos q\phi_\lambda + \ldots\} a\rho_\lambda$$

$$+ \frac{m'}{a'^2}\left\{\frac{\partial}{\partial \alpha}(b_1 \sin\phi_\lambda + \ldots + qb_q \sin q\phi_\lambda + \ldots) - \sin\phi_\lambda\right\}\sigma\lambda,$$

and

$$\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)_0 + a\rho_\lambda \left\{\frac{\partial}{\partial r_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0 + \sigma_\lambda \left\{\frac{\partial}{\partial \theta_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0$$

$$= \frac{m'}{aa'}(\ldots + qb_q \sin q\phi_\lambda + \ldots) - \frac{m'}{a'^2}\sin\phi_\lambda$$

$$+ \left[\frac{m'}{a'^2 a}\frac{\partial}{\partial \alpha}(b_1 \sin\phi_\lambda + \ldots + qb_q \sin q\phi_\lambda + \ldots)\right.$$

$$\left. - \frac{m'}{a'a^2}(b_1 \sin\phi_\lambda + \ldots + qb_q \sin q\phi_\lambda + \ldots)\right] a\rho_\lambda$$

$$+ \left[-\frac{m'}{aa'}(\ldots + q^2 b_q \cos q\phi_\lambda + \ldots) + \frac{m'}{a'^2}\cos\phi_\lambda\right]\sigma_\lambda.$$

* The work is not reproduced in detail for it is to be found in GOLDSBROUGH I, pp. 103–106; here the notation is slightly different from that of GOLDSBROUGH.

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Inserting the above values in the equations of the perturbed motion of the particle λ, and remembering that $\left(\frac{\partial F_\lambda}{\partial r_\lambda}\right)_0 = -a\omega^2$, $\left(\frac{1}{r_\lambda}\frac{\partial F_\lambda}{\partial \theta_\lambda}\right)_0 = 0$, and dividing both sides by a, we obtain the equations of the perturbed motion of the particle λ in the following form :—

$$\begin{aligned}
\ddot{\rho}_\lambda - 2\omega\,\dot{\sigma}_\lambda = &\left[3\omega^2 + \frac{2m}{a^3} - \frac{m}{8a^3}\sum_\mu{}' \left\{\frac{1}{\sin^3(\mu\sim\lambda)\,\pi/n} + \frac{1}{\sin(\mu\sim\lambda)\,\pi/n}\right\}\right]\rho_\lambda \\
&+ \frac{m}{8a^3}\sum_\mu{}'\left[\frac{1+\sin^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + 2\cos(\mu-\lambda)\,2\pi/n\right]\rho_\mu \\
&+ \frac{m}{8a^3}\sum_\mu{}'\left[\frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n} + 1\right]\sin(\mu-\lambda)\,2\pi/n\,\sigma_\mu \\
&+ \frac{m'}{aa'^2}\left[\frac{\partial}{\partial\alpha}\left(\tfrac{1}{2}b_0 + b_1\cos\phi_\lambda + \ldots + b_q\cos q\phi_\lambda + \ldots\right) - \cos\phi_\lambda\right] \\
&+ \frac{m'}{a'^3}\left\{\frac{\partial^2}{\partial\alpha^2}\left(\tfrac{1}{2}b_0 + b_1\cos\phi_\lambda + \ldots + b_q\cos q\phi_\lambda + \ldots\right)\right\}\rho_\lambda \\
&+ \frac{m'}{aa'^2}\left\{\frac{\partial}{\partial\alpha}\left(b_1\sin\phi_\lambda + \ldots + qb_q\sin q\phi_\lambda + \ldots\right) - \sin\phi_\lambda\right\}\sigma_\lambda \quad . \qquad (\mathrm{E}.\lambda)
\end{aligned}$$

$$\begin{aligned}
\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda = &\frac{m}{a^3}\sum_\mu{}'\left[2 - \frac{1}{16\sin^3(\mu\sim\lambda)\,\pi/n}\right]\sin(\mu-\lambda)\,2\pi/n\,\rho_\mu \\
&- \frac{m}{a^3}\sum_\mu{}'\left[\frac{1+\cos^2(\mu\sim\lambda)\,\pi/n}{8\sin^3(\mu\sim\lambda)\,\pi/n} + \cos(\mu-\lambda)\,2\pi/n\right](\sigma_\mu - \sigma_\lambda) \\
&+ \frac{m'}{aa'}\left[\frac{1}{a}\left(\ldots + qb_q\sin q\phi_\lambda + \ldots\right) - \frac{1}{a'}\sin\phi_\lambda\right] \\
&+ \frac{m'}{aa'}\left[\frac{1}{a'}\frac{\partial}{\partial\alpha}\left(b_1\sin\phi_\lambda + \ldots + qb_q\sin q\phi_\lambda + \ldots\right)\right. \\
&\qquad\qquad \left. - \frac{1}{a}\left(b_1\sin\phi_\lambda + \ldots + qb_q\sin q\phi_\lambda + \ldots\right)\right]\rho_\lambda \\
&+ \frac{m'}{aa'}\left[\frac{1}{a'}\cos\phi_\lambda - \frac{1}{a}\left(\ldots + q^2b_q\cos q\phi_\lambda + \ldots\right)\right]\sigma_\lambda. \quad . \ . \ . \ . \ . \qquad (\mathrm{G}.\lambda)
\end{aligned}$$

The equations (E.λ) and (G.λ) describe the perturbed motion of the typical particle λ.

If $m' = 0$, the equations (E.λ) and (G.λ) reduce to equations (A.λ) and (B.λ) of § 2.

As in § 2 we have n interdependent pairs of linear differential equations of the second order (E.λ) and (G.λ) similar to the n pairs (A.λ) and (B.λ). There are, however, two important differences between the two sets of n pairs :—

(1) There are no independent terms in the equations (A.λ) and (B.λ).

(2) The coefficients of ρ_λ and σ_λ in (E.λ) and (G.λ) are not constants, but known functions of t.

It was possible to derive n independent pairs (A$'.s$) and (B$'.s$) of linear differential equations for the "secondary" variables by employing the transformation introduced in § 3. It is easy to see, from § 3, that we shall not be able to derive n independent pairs of differential equations from the n interdependent pairs of differential equations (E.λ) and (G.λ) by employing the transformation contained in § 3 solely on account of the ϕ_λ which occurs in terms factored by m' in the equations (E.λ) and (G.λ).

The writer does not know how one can derive n independent pairs of linear differential equations of the second order from the n interdependent pairs (E.λ) and (G.λ). Consequently, it is not possible to solve the n pairs (E.λ) and (G.λ), even theoretically, at least at present.

GOLDSBROUGH's* method of solving the equations (E.λ) and (G.λ) will now be described and it will be shown that his method is not valid.

GOLDSBROUGH (I, p. 106) introduced the hypothesis that $\rho_{\lambda+1} = \beta\rho_\lambda$, and also, tacitly, $\sigma_{\lambda+1} = \beta\sigma_\lambda$.

Then according to GOLDSBROUGH,

$$\rho_\lambda = \rho_{n+\lambda} = \beta^n\rho_\lambda, \qquad \sigma_\lambda = \sigma_{n+\lambda} = \beta^n\sigma_\lambda.$$

Therefore,

$$\beta^n = 1 \text{ or } \beta = \cos\frac{2\pi s}{n} + i\sin\frac{2\pi s}{n},$$

where

$$i = \sqrt{-1} \text{ and } s = 0, 1, \dots n-1 \text{ (or } s = 1, \dots, n).$$

Considering some fixed value of s,

$$\rho_{\lambda+1} = e^{\frac{2\pi i s}{n}}\rho_\lambda, \qquad \sigma_{\lambda+1} = e^{\frac{2\pi i s}{n}}\sigma_\lambda$$

or

$$\rho_\lambda = \rho_\mu e^{\frac{2\pi i(\lambda-\mu)s}{n}}, \qquad \sigma_\lambda = \sigma_\mu e^{\frac{2\pi i(\lambda-\mu)s}{n}},$$

i.e.,

$$\frac{\rho_\lambda}{e^{\frac{2\pi i\lambda s}{n}}} = \frac{\rho_\mu}{e^{\frac{2\pi i\mu s}{n}}} = X_s \text{ (say)}, \qquad \frac{\sigma_\lambda}{e^{\frac{2\pi i\lambda s}{n}}} = \frac{\sigma_\mu}{e^{\frac{2\pi i\lambda s}{n}}} = Y_s \text{ (say)}.$$

Thus there are two fundamental quantities which, for the sake of convenience, have been called X_s, Y_s, corresponding to every value of s. Owing to the relations $\rho_\lambda = X_s e^{\frac{2\pi i\lambda s}{n}}$, $\sigma_\lambda = Y_s e^{\frac{2\pi i\lambda s}{n}}$ ($s = 0, 1, \dots, n-1$ or $1, \dots, n$), each pair X_s, Y_s has to satisfy n pairs of differential equations when these formulæ for ρ_λ and σ_λ are used in the equations (E.λ) and G.λ). However, each particle has its characteristic ϕ : $\phi_\mu - \phi_\lambda = \dfrac{2\pi(\lambda-\mu)}{n}$ and ϕ_λ does play an important part

* Hitherto, in this section, the notation is substantially the same as GOLDSBROUGH I (pp. 103–106). $a\rho_\lambda$ has been used instead of GOLDSBROUGH's ρ_λ. GOLDSBROUGH started with ρ_λ and virtually changed it later to $a\rho_\lambda$ (I, p. 107).

in the equations (E.λ) and (G.λ). Consequently, corresponding to any value of s the hypothesis $\rho_{\lambda+1} = \beta\,\rho_{\lambda}$, $\sigma_{\lambda+1} = \beta\,\sigma_{\lambda}$ introduces two new dependent variables X_s, Y_s (say) which have to satisfy n pairs of linear differential equations in which the coefficients of X_s and Y_s vary from pair to pair. The simplifying hypothesis leads, therefore, to an impasse. It is to be noticed that such a situation will not arise in equations (A.λ) and (B.λ) to which the equations (E.λ) and (G.λ) reduce if we put $m' = 0$. If we put $\rho_{\lambda} = X_s\, e^{\frac{2\pi i\lambda s}{n}}$, $\sigma_{\lambda} = Y_s\, e^{\frac{2\pi i\lambda s}{n}}$ $(s = 1, \ldots, n)$ in the equations (A.λ) and (B.λ) the n pairs of equations satisfied by X_s and Y_s for any fixed value of s all reduce to a single pair of equations, for X_s and Y_s, which is identical with the pair (A′.s) and (B′.s). It may be remarked, therefore, that GOLDSBROUGH's hypothesis is embodied in the transformation formulæ (3.1/λ) and (3·2/λ), the X_s's corresponding to the k_s's and the Y_s's corresponding to the l_s's $(s = 1, \ldots, n)$.*

After using this anomalous hypothesis, *i.e.*, after putting $\rho_{\mu} = e^{\frac{2\pi i\,(\mu-\lambda)\,s}{n}}\,\rho_{\lambda}$ and $\sigma_{\mu} = e^{\frac{2\pi i\,(\mu-\lambda)\,s}{n}}\,\sigma_{\lambda}$ in the equations (E.λ) and (G.λ), and dropping the suffix λ in ϕ_{λ}, one reaches equations corresponding to equations (3) of GOLDSBROUGH (I, p. 107). GOLDSBROUGH based his investigation, of the perturbations on a ring of equal masses, on these equations, which do not exist. He obtained their solutions in § 3 of his first paper. A division corresponded to large values of "ρ" and "σ" caused by the vanishing of denominators in the formulæ for "ρ" and "σ" thus obtained.

After examining, in the manner explained above, the effects due to a number of satellites, one at a time, GOLDSBROUGH attributed certain divisions to certain satellites.

I wish to thank Professor H. F. BAKER, Professor G. R. GOLDSBROUGH, and Dr. W. M. SMART for their great interest in my work.

SUMMARY

In Part I (§§1–12) the stability of the steady motion of a collection of n particles of equal mass about the centre of a homogeneous sphere is discussed on the assumption that the collection of particles and the sphere form an "isolated" system and that the motion of every member of the whole system relative to Newtonian axes is described by the law of gravitation. In the steady state the particles are assumed to form the vertices of a regular polygon of n sides, inscribed in a circle centre O and radius a, rotating with uniform angular velocity ω. The steady state is dynamically possible without any approximations provided ω is suitably chosen.

* The foregoing observations on the hypothesis $\rho_{\lambda+1} = \beta\rho_{\lambda}$ and $\sigma_{\lambda+1} = \beta\sigma_{\lambda}$ were communicated to Professor G. R. GOLDSBROUGH. In a letter to the writer, dated June 8, 1933, Professor GOLDSBROUGH acknowledged that the simplifying hypothesis did not serve its purpose.

The system of particles is then assumed to undergo a slight disturbance in the initial plane of the system ; the equations (A.λ) and (B.λ) describe the disturbed motion of the typical particle λ as far as quantities of the first order of smallness are concerned. For the whole collection of particles we obtain n interdependent pairs of linear differential equations with constant coefficients for the $2n$ dependent variables ρ_λ's and σ_λ's. "Secondary" dependent variables k_s and l_s ($s = 1, \ldots, n$) are introduced in § 3 and it is shown that the "secondary" dependent variables satisfy n independent pairs (A'.s) and (B'.s) of linear differential equations of the second order with constant coefficients.

n "stability" equations (C.s) are obtained in § 6, and the condition for stability is that all these equations have real roots.

The steady motion is unstable when $n = 2$, the particles being diametrically opposite, and the instability is independent of a (§ 8).

The steady motion is also unstable when $n = 3, 4, 5, 6$ (§ 10).

It is not possible to make general statements about stability as the exact values of some quantities are unknown (§§ 7 and 10).

Hence we conclude that there is a lower limit to the value of n in order that the steady motion be stable.

Maxwell's limit for stability is investigated in § 11 ; the difference between the limit obtained in this paper and the limit obtained by Maxwell is quite appreciable.

The whole problem of the divisions in the system of rings and the possible role played by the satellites of Saturn in their formation remain (Part II).